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FROM CAGNIARD'S METHOD FOR SOLVING SEISMIC PULSE PROBLEMS TO THE METHOD OF THE DIFFERENTIAL TRANSFORM

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Abstract—Cagniard's method is well-known in mathematical geophysics. The integral transform method of Cagniard involves path deformations in the complex plane, residue calculus and, in some cases, branch cuts. The technique of the differential transform, which evolved from that of Cagniard, involves an integral-free transform and avoids the integral considerations associated with Cagniard's technique. These two techniques are examined here by considering their application to Garvin's problem, a classical pulse-propagation problem in geophysics. This examination demonstrates the superiority of the technique of the differential transform over that of Cagniard, thus confounding the belief that the intricate integral considerations involved in Cagniard's technique are unavoidable.

1. INTRODUCTION

In this article we consider (a) a typical pulse propagation problem in seismology; (b) a classical method, Cagniard's method, for solving it; and (c) a new method, that of the differential transform, that renders the classical analysis simpler and more attractive.

The determination of the effects of a free surface on the propagation of elastic pulses is a basic problem in theoretical seismology. Prior to Garvin's classical paper[1], most investigators used methods consisting of a formal solution to a steady-state problem which was then generalized to a pulse by means of a Fourier integral. The resulting double integrals could be handled only by considering large distances from the pulse source[1].

Garvin[1] tackled a two-dimensional pulse problem using Cagniard's method. Garvin obtained an exact, closed, algebraic expression for the displacement of a point on the surface of a solid, excited by a pulse, as a function of time. Expressions for interior points were later provided by De Hoop[2], for a surface impulsive line source, and by Alterman and Loeventhal[3], for a buried impulsive line source. The achievement of Garvin in obtaining an exact, closed, algebraic solution to an interesting seismic pulse-propagation problem thus demonstrates the power of Cagniard's method. Cagniard himself applied his method to a point pulse problem[4], which is more involved than the line pulse problem of Garvin, and can hardly be used to illustrate his method[5].

In order to illustrate both Cagniard's method and that of the differential transform, Garvin's problem and its solution are considered here by means of the two methods. The advantage of the differential transform method over that of Cagniard will then become evident. Here by Cagniard's method we mean the method described in Sections 3 and 6. We do not treat other versions of the method found in the literature such as modifications of Cagniard's method due to De Hoop[6] and Gakenheimer and Miklowitz[7].

The purpose of this article is not to produce a new solution but to illustrate the use of the differential transform technique for solving seismic pulse problems. In particular, it is intended to demonstrate that integral considerations involved in Cagniard's method are unnecessary. Integral considerations, such as residue calculus and deformation of paths of complex contour integrals, which are involved in Cagniard's method and hence in Garvin's work, are misleading. They do not form a natural part of the study of those geophysical problems for which Cagniard's method was developed and it is shown that it is not necessary to deduce physical results from them.

For example, an argument justifying the choice of the right solution in Cagniard's method rests on the contour of integration employed; changing a variable of integration results in a number of possible contours which lead to several possible solutions. The right unique solution to the physical problem is selected by retaining one appropriate contour of integration and rejecting the others. However, since integral considerations are avoidable, as shown in this article, it is not necessary to base on them the choice of the right solution. In the present article, the choice of the right solution is based on the requirement that the solution should introduce no new singularities into the medium under consideration; the only singularity allowed is that due to the pulse source function.

Using either the differential transform method or Cagniard's method, a problem of choosing the right solution arises; a change of variable requires the solution of a quartic equation, leading to two non-identical solutions, one of which must be rejected. Since there are no integral considerations involved in the technique of the differential transform, integral considerations are not used for accepting one solution in preference to another. Indeed, it was found that one of the two distinct solutions introduce new singularities into the elastic medium under consideration and therefore must be rejected. In other words, the wrong solution is rejected by means of a physical argument. Hence although Cagniard's method, as described in Section 6, leads to the correct result, the result is obscured by unnecessary arguments involving integral considerations.

The different types of elastic waves observed in solutions by Cagniard's method arise from different poles and branch cuts in the complex plane. It is therefore believed that poles and branch cuts play a decisive role in obtaining theoretical seismograms by Cagniard's method. This present work thus breaks that belief; in the commutative Diagram 6.1 the starting point and the destination are connected by two paths, representing Cagniard's technique and the differential transform technique. Although both paths lead to the same destination, i.e. both techniques yield the same result, one of them involves a complex contour integral while the other one is free of integrals.

The possibility of employing the differential transform to solve problems which are traditionally attacked by integral transforms is due to the fact that the differential transform is equivalent to a formal integral transform in which the formal integrals are eliminated. The eliminated integrals are formal in the sense that there is no need to insure their existence. The presentation of the differential transform as an integral-free integral transform is given in [8, 9] and a simple application for solving a boundary value problem is given in [19].

2. THEORY

Since both Cagniard's method and the method of the differential transform are exemplified, in the present paper, by employing them to solve an elastic wave propagation problem, i.e. Garvin's problem [1], the relavent theory of two-dimensional elastic waves is given here.

Following ([10], chap. 2), as there is no dependance in the y direction, the displacement in the xz-plane of any point is (u, w), where u and w are the components of the displacement vector in the x and z directions respectively. The displacement components u and w can be expressed in terms of potentials ϕ and ψ,

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \qquad w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}, \tag{2.1}$$

where the potentials ϕ and ψ are respective solutions to the two wave equations

$$\nabla^2 \phi = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \qquad \nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2}, \tag{2.2}$$

∇^2 being the two-dimensional Laplacian operator,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

In equations (2.2) α and β, the speeds of the P and S waves of elasticity, are given by

$$\alpha = \left[\frac{\lambda + 2\mu}{\rho}\right]^{1/2} \qquad \beta = \left[\frac{\mu}{\rho}\right]^{1/2}, \tag{2.3}$$

where λ and μ are Lame's constants for the solid and ρ is the density of the solid. Thus the problem of determining u and w is reduced to one of finding solutions ϕ and ψ of equations (2.2) and associated boundary conditions. For a free surface at $z = 0$, the normal and shearing stresses P_{zz} and P_{xz} must vanish. The boundary conditions for $z = 0$, in terms of the potentials are accordingly

$$\lambda \nabla^2 \phi + 2\mu \left[\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z}\right] = 0 \qquad \mu\left[2\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2}\right] = 0. \tag{2.4}$$

Based on the theory of two-dimensional elastic waves of this section, Garvin's source function is described in Section 3 and Garvin's problem is presented in Section 4.

3. GARVIN'S SOURCE FUNCTION

A source function associated with two-dimensional elastic waves is a solution of the elastic wave equations (2.2) in an unbounded solid which gives rise to a displacement field that is singular at a point and tends to zero at infinity. Consider the reduced wave equation

$$\nabla^2 \tilde{\phi} = \frac{s^2}{\alpha^2}\tilde{\phi} \tag{3.1}$$

which is the Laplace transform of the first wave equation in (2.2),

$$\nabla^2 \phi = \frac{1}{\alpha^2}\frac{\partial^2 \phi}{\partial t^2}. \tag{3.2}$$

Under some obvious restrictions on ϕ, if $\phi(x, z, t)$ satisfies the wave equation (3.2) then its Laplace transform $\tilde{\phi}(x, z, s)$ satisfies the reduced wave equation (3.1). An actual solution of the actual problem (3.2) is therefore the inverse Laplace transform of an operational solution of the operational problem (3.1). Cagniard's method, as applied to Garvin's source function, relies on the fact that a simple parametric operational solution $\tilde{\phi}$,

$$\tilde{\phi} = s^{-1}\, e^{-s/\alpha(x \cosh \tau + iz \sinh \tau)} \tag{3.3}$$

of the reduced wave equation (3.1), may be transformed into an actual solution (3.4) of the wave equation (3.2),

$$\phi = \begin{cases} \cosh^{-1}\left(\dfrac{\alpha t}{r}\right), & t > \dfrac{r}{\alpha} \\ 0, & t < \dfrac{r}{\alpha} \end{cases}, \quad r^2 = x^2 + z^2, \tag{3.4}$$

or equivalently

$$\phi = \begin{cases} \partial_t^{-1}\left(t^2 - \dfrac{r^2}{\alpha^2}\right)^{-1/2}, & t > \dfrac{r}{\alpha} \\ 0, & t < \dfrac{r}{\alpha}. \end{cases} \tag{3.5}$$

The transformation of (3.3) into (3.5) is illustrated in Diagram 3.1 below. The solution ϕ in (3.4) and (3.5) forms a convenient representation for a line source of an out-going pulse. The line source function (3.4) was used by Garvin in his study of the reflection of elastic pulses emitted from an impulsive line source[1]. The operator ∂_t^{-1} signifies anti-differentiation with respect to t. The constant of integration in the application of ∂_t^{-1} need not be specified since differentiation with respect to t is performed when the displacements are calculated, i.e. the operator ∂_t^{-1} will always be subjected to its inverse operator ∂_t in a later step.

The solution $\tilde{\phi}$ of (3.3) is simpler than the solution ϕ of (3.5) in the sense that $\tilde{\phi}$ involves two separated variables x and z, while ϕ involves three variables x, z and t which are not separated. It should be noticed that any function of the parameter τ is regarded as a constant with respect to the reduced wave equation (3.1). Thus the expression

$$x\cosh\tau + iz\sinh\tau$$

in (3.3) is a linear combination of x and z. The solution $\tilde{\phi}$ in (3.3) has, therefore, the simple form of the exponential function of a linear combination of the variables x and z multiplied by a constant.

The required transformation from the operational solution $\tilde{\phi}$ of (3.1) to an actual solution of (3.2) is shown in Diagram 3.1 below, where

$$r^2 = x^2 + z^2 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{z}{x}\right).$$

$$(1) \quad s^{-1} e^{-\frac{s}{\alpha}(x\cosh\tau + iz\sinh\tau)}$$

$$\Big\downarrow \int d\tau$$

$$(2) \quad s^{-1}\int_0^\infty e^{-\frac{s}{\alpha}(x\cosh\tau + iz\sinh\tau)}\,d\tau$$

$$\Big\downarrow \mathrm{Re}\,L^{-1}$$

$$(3) \quad \mathrm{Re}\,L^{-1} s^{-1}\int_0^\infty e^{-\frac{s}{\alpha}(x\cosh\tau + iz\sinh\tau)}\,d\tau$$

$$= \mathrm{Re}\,L^{-1} s^{-1}\int_0^\infty e^{-\frac{sr}{\alpha}(\cosh(\tau+i\theta))}\,d\tau$$

$$\Big\downarrow =$$

$$(4) \quad \mathrm{Re}\,L^{-1} s^{-1}\int_{i\theta}^{\infty+i\theta} e^{-\frac{sr}{\alpha}\cosh\tau}\,d\tau$$

$$= \mathrm{Re}\,L^{-1} s^{-1}\left[\int_0^\infty e^{-\frac{sr}{\alpha}\cosh\tau}\,d\tau - \int_0^{i\theta} e^{-\frac{sr}{\alpha}\cosh\tau}\,d\tau\right]$$

$$= \mathrm{Re}\,L^{-1} s^{-1}\left[\int_0^\infty e^{-\frac{sr}{\alpha}\cosh\tau}\,d\tau - i\int_0^{\theta} e^{-\frac{sr}{\alpha}\cos\tau}\,d\tau\right]$$

$$= L^{-1} s^{-1}\int_0^\infty e^{-\frac{sr}{\alpha}\cosh\tau}\,d\tau$$

$$\Big\downarrow$$

$$(5) \quad L^{-1} s^{-1}\int_{\frac{r}{\alpha}}^\infty \left(t^2 - \frac{r^2}{\alpha^2}\right)^{-\frac{1}{2}} e^{-st}\,dt$$

$$\Big\downarrow =$$

$$(6) \quad \partial_t^{-1}\left(t^2 - \frac{r^2}{\alpha^2}\right)^{-\frac{1}{2}} = \cosh^{-1}\left(\frac{\alpha t}{r}\right), \quad t > \frac{r}{\alpha}$$

Diagram 3.1.

The starting point (1) of Diagram 3.1 is a parametric operational solution of the reduced wave equation (3.1) and the destination (6) is a singular solution to the wave equation (3.2). This singular solution is Garvin's source function and the method of obtaining it, as described in Diagram 3.1, is Cagniard's method.

The parametric operational solution (1) is integrated over the parameter τ to obtain another operational solution in (2). The integral in (2) is destined to play against the inverse Laplace transform operator L^{-1} to the point of their mutual annihilation in (6). A solution of the wave equation (3.2) is obtained at point (3) of Diagram 3.1 by applying the real part of the inverse Laplace transform to the operational solution in (2). The crucial step of Cagniard's method, an appropriate change of variable in (3), gives a complex contour integral in (4), the contour of which is deformed to that shown in (5). The solution in (5) has a form which enables the Laplace transform inversion to be done by *inspection*, thus obtaining the solution in (6) which is the destination point of Diagram 3.1.

It should be noted here that in a more general case the exponential functions of Diagram 3.1 may be multiplied by any function of the parameter τ, as is the case in the Laplace transform inversions of the next section.

Diagram 3.1 exhibits Cagniard's method as a method of throwing an operational solution into a form that enables the Laplace transform inversion to be done by *inspection*. The appropriate deformation of the complex contours associated with Cagniard's method, an example of which is provided in Diagram 3.1, may involve considerations of residues and branch cuts[11]. Moreover, there are some modifications to Cagniard's method due to De Hoop[6] and to Gakenheimer and Miklowitz[7]. It is therefore clear that there is a variety of ways to implement the idea of Cagniard's method.

Garvin's source function, at the destination point of Diagram 3.1, was obtained, by Garvin, using a deformation of path of integration which is different from that used in Diagram 3.1. The path deformation of Diagram 3.1 was developed by Robinson and Ungar[12].

4. GARVIN'S PROBLEM AND SOLUTION

Let a line source be situated at $(0, y, h)$ in a semi-infinite solid occupying the semi-infinite space $z > 0$ of the (x, y, z)-space with Cartesian coordinates x, y and z. The solid is bounded by the free surface $z = 0$ on which the boundary conditions (2.4) must be satisfied. For his source function ϕ_0, equation (3.4),

$$\phi_0 = \begin{cases} \cosh^{-1}\dfrac{\alpha t}{r}, & t > \dfrac{r}{\alpha} \\ 0, & t < \dfrac{r}{\alpha} \end{cases}, \quad r^2 = x^2 + (z-h)^2,$$

Garvin used the integral representation ([1], p. 535)

$$\phi_0 = ReL^{-1}s^{-1}\int_0^\infty e^{-s/\alpha(x\cosh\tau + i(z-h)\sinh\tau)}\,d\tau, \tag{4.1}$$

the proof of which was not supplied by Garvin, but is given in Diagram 3.1.

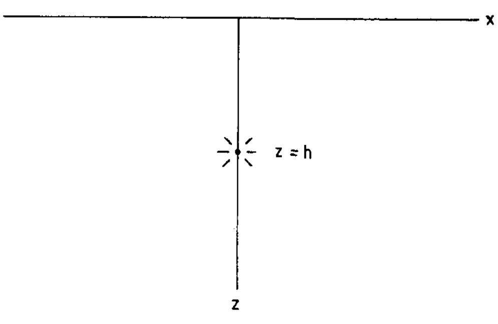

Fig. 4.1. The geometry of Garvin's problem with point source at $(0, h)$.

The nature of Garvin's problem suggests that a solution G involves direct and reflected waves and thus has the form

$$G(x, y, z) = \phi_0 + \phi_1 + \psi_1, \tag{4.2}$$

in which ϕ_1 and ψ_1 are the reflected P and S-waves given by

$$\phi_1 = ReL^{-1}s^{-1}\int_0^\infty A(\tau)\, e^{-s/\alpha(x\cosh\tau - iz\sinh\tau)}\, d\tau$$

$$\psi_1 = ReL^{-1}s^{-1}\int_0^\infty B(\tau)\, e^{-s/\alpha(x\cosh\tau - izC(\tau))}\, d\tau. \tag{4.3}$$

The functions A, B and C in equations (4.3) are determined by the boundary conditions (2.4) and the second wave equation in (2.2).

The form of the solution to Garvin's problem in equations (4.2) and (4.3) is known from the theory of plane elastic waves; the simple structure of plane waves allows a simple solution. It is not necessary to evaluate the integrals in (4.3) in order to find the solution (4.2). An integral free solution to Garvin's problem is obtained by performing the Laplace transform inversions in (4.3) by means of Cagniard's method, as illustrated in Diagram 3.1. The crucial point, however, is that it might be difficult or impossible to follow the procedure in Diagram 3.1 if complicated functions $A(\tau)$, $B(\tau)$ or $C(\tau)$ are inserted under the integral signs of (4.3). If the functions $A(\tau)$ and $B(\tau)$ are too involved, it is difficult to locate their singularities when a path deformation is required. This diffiiculty is completely removed in Section 7, with the application of the differential transform of Section 6.

Diagram 3.1, used for obtaining Garvin's representation (4.1), gives the impression that the transformation from an operational solution to an actual one cannot be achieved without integral considerations. Analysing Diagram 3.1, in terms of a new notation for functions, however, reveals that integral considerations can be eliminated. The new notation, presented in the next section, will enable us, in Section 6, to display Diagram 3.1 in a simpler and more general form in which it becomes apparent that the integral considerations of Diagram 3.1 are avoidable.

5. THE CHANGE OF VARIABLE FORMULA

In Diagram 3.1 the change of variable plays a decisive role; it throws an integral into a form which allows the Laplace transform inversion to be done by inspection. In order to formalize the features of Diagram 3.1 for general use, a convenient change of variable formula is needed. Following[8] the new asterisk notation for functions is used for expressing the change of variable formula (5.1) below. In this notation a function with value y is denoted by y^*, and the change of variable formula, also known as integration by substitution, takes the form[8]

$$\int F(\tau; t^*(x;\tau))\, d\tau = \int F(\tau^*(x;t); t)\dot{\tau}^*\, dt. \tag{5.1}$$

In equation (5.1) t^* is a function of the variable of integration τ, and of the n variables $x = (x_1, x_2, \ldots, x_n)$, assuming the value t,

$$t = t^*(x; \tau). \tag{5.2}$$

The transformation from the variable of integration τ to the new variable of integration t in equation (5.1), is achieved by means of the substitution (5.2); the function $t^*(x;\tau)$ of the old variable of integration τ becomes the new variable of integration t. As a result, the old variable of integration τ becomes a function τ^* of the new variable of integration t. Thus τ^* in (5.1) is a function of the new variable of integration t, and of x, assuming the value τ,

$$\tau = \tau^*(x; t), \tag{5.3}$$

the derivative of which with respect to t is denoted by $\dot{\tau}^*(x, t)$, or by $\dot{\tau}^*$ when the content of the argument is clear,

$$\dot{\tau}^* = \frac{\partial \tau^*}{\partial t}(x;t). \tag{5.4}$$

There is a connection between the functions t^* and τ^* in the change of variable formula (5.1); these functions are inverse to one another in the sense that

$$t^*(x;\tau^*(x;t)) = t \quad \text{and} \quad \tau^*(x;t^*(x;\tau)) = \tau \tag{5.5}$$

are identities for x and t in a domain.

6. CAGNIARD'S METHOD AND THE DIFFERENTIAL TRANSFORM

The change of variable formula (5.1) allows a more general form of Diagram 3.1 seen in Diagram 6.1. Diagram 3.1 deals with a specific function and is a particular case of Diagram 6.1.

$$\begin{tikzcd}
(1)\ F(x;\tau)\, e^{-st^*(x;\tau)} \arrow[r, "\int d\tau"] \arrow[ddd] & (2)\ \int_\Gamma F(x;\tau)\, e^{-st^*(x;\tau)}\, d\tau \arrow[d, "L^{-1}"'] \\
& (3)\ L^{-1} \int_\Gamma F(x;\tau)\, e^{-st^*(x;\tau)}\, d\tau \arrow[d, "="'] \\
& (4)\ L^{-1} \int_{\Gamma'} F(x;\tau^*)t^*\, e^{-st}\, dt \arrow[d, "="'] \\
(6)\ F(x;\tau^*)t^* & (5)\ L^{-1} \int_0^\infty F(x;\tau^*)t^*\, e^{-st}\, dt \arrow[l, "="']
\end{tikzcd}$$

Diagram 6.1.

The usefulness of the asterisk notation is clearly evident in Diagram 6.1. The expression in (1) is integrated over an unspecified contour Γ which is replaced by Γ' in (4), due to the change of variable of integration. The change of variable in (4) is accomplished by formula (5.1). Γ is specified at point (5) to be such that Γ' of (4) is the positive ray, thus allowing the Laplace transform inversion to be done by inspection. One general point that emerges is that a direct path from point (1) to point (6) may be allowed, thus bypassing the integral considerations. The operator that transforms expression (1) directly into expression (6) in Diagram 6.1 is the differential transform U, defined in equation (6.1) below.

The differential transform U is defined by [8]

$$U\{s^n F(x;\tau)\, e^{-st^*(x;\tau)}\} = \partial_t^n\{F(x;\tau^*(x;t))\dot{\tau}^*(x;t)\} \tag{6.1}$$

where

(a) $t^*(x;\tau)$, the *determining function*, is a suitably differentiable function of the real or complex set of variables $x = (x_1, x_2, \ldots, x_n)$, and of the *inversion parameter* τ, but is otherwise arbitrary.

(b) $\tau^*(x;t)$ is the inverse of the determining function $t^*(x;\tau)$.

(c) $\dot{\tau}^* = \partial\tau^*/\partial t$ is the derivative of τ^* with respect to the *determining parameter t.*

The integer n in (6.1) is normally non-negative. However, it can be negative and the operator ∂_t^{-1} associated with s^{-1} assigns anti-differentiation with an unspecified constant of integration, as in Diagram 3.1.

Two fundamental properties of the differential transform are

$$U\partial_{x_i} = \partial_{x_i} U, \qquad i = 1, 2, \ldots, n$$
$$Us = \frac{\partial}{\partial t} U, \tag{6.2}$$

and from its definition (6.1), $U\{0\} = 0$, details of which are given in [8, 9].

Due to its properties in (6.2), it is clear that the differential transform U sends a solution of the reduced wave equation (3.1) into a solution of the wave equation (3.2): Let $\tilde{\phi}$ satisfy (3.1) and have an appropriate form for the application of U, and let $\phi = U\{\tilde{\phi}\}$, then, by (6.2),

$$\begin{aligned}\left(\partial_x^2 + \partial_z^2 - \frac{1}{\alpha^2}\partial_t^2\right)\{\phi\} &= \left(\partial_x^2 + \partial_z^2 - \frac{1}{\alpha^2}\partial_t^2\right) U\{\tilde{\phi}\} \\ &= U\left\{\left(\partial_x^2 + \partial_z^2 - \frac{s^2}{\alpha^2}\right)\tilde{\phi}\right\} \\ &= U\{0\} = 0.\end{aligned}$$

It is now clear from Diagram 6.1 that the differential transform method can be used instead of that of Cagniard's method. An example of a simple boundary value problem solved by means of the differential transform is given in [19]. Further examples are given in [13, 14].

7. THE USE OF THE DIFFERENTIAL TRANSFORM

In this section we shall demonstrate the use of the differential transform by presenting a function, $\tilde{\phi}$, which under the application of the differential transform becomes Garvin's source ϕ. The function $\tilde{\phi}$ will later be used, in Section 8, for solving Garvin's problem.

Let

$$t^*(x, z, \tau) = \frac{1}{\alpha}(x \cosh \tau + i(z-h)\sinh \tau) \tag{7.1}$$

be a determining function in the differential transform definition (6.1). The inverse τ^* of t^* is obtained by solving the equation

$$\alpha t = x \cosh \tau^* + i(z-h)\sinh \tau^* \tag{7.2}$$

for τ^*. Equation (7.2) is obtained from equation (7.1) by removing the asterisk from t^* and imposing it on τ. In polar coordinates,

$$x = r\cos\theta, \qquad z - h = r\sin\theta$$

equation (7.2) may be written as

$$\alpha t = r\cosh(\tau^* + i\theta)$$

from which τ^* may be found explicitly as

$$\tau^*(x, z, t) = \cosh^{-1}\left(\frac{\alpha t}{r}\right) - i\theta \tag{7.3}$$

where

$$r^2 = x^2 + (z-h)^2 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{z-h}{r}\right).$$

The derivative of τ^* with respect to t is $\dot\tau^*$,

$$\dot\tau^* = \frac{\partial\tau^*}{\partial t} = \left(t^2 - \frac{r^2}{\alpha^2}\right)^{-1/2}. \tag{7.4}$$

The function

$$\tilde\phi = s^{-1}\, \mathrm{e}^{-st^*(x,\,z,\,\tau)} \tag{7.5}$$

has a proper form for the application of the differential transform. Letting ϕ signify the real part of the differential transform of $\tilde\phi$, we have

$$\begin{aligned}\phi &= ReU\{\tilde\phi\} = ReU\{s^{-1}\,\mathrm{e}^{-s/\alpha(x\cosh\tau + i(z-h)\sinh\tau)}\}\\ &= ReU\{s^{-1}\,\mathrm{e}^{-st^*}\} = Re\partial_t^{-1}\dot\tau^* = Re\partial_t^{-1}\left(t^2-\frac{r^2}{\alpha^2}\right)^{-1/2}\\ &= \begin{cases}\cosh^{-1}\left(\dfrac{\alpha t}{r}\right), & |t| > \dfrac{r}{\alpha}\\ 0, & |t| < \dfrac{r}{\alpha}\end{cases}\end{aligned} \tag{7.6}$$

which is Garvin's source function.

Comparing the differential transform representation (7.6),

$$\cosh^{-1}\left(\frac{\alpha t}{r}\right) = ReU\{s^{-1}\,\mathrm{e}^{-s/\alpha(x\cosh\tau+i(z-h)\sinh\tau)}\}, \qquad t > \frac{r}{\alpha} \tag{7.7}$$

with the integral transform representation given in equation (4.1),

$$\cosh^{-1}\left(\frac{\alpha t}{r}\right) = ReL^{-1}\int_0^\infty s^{-1}\,\mathrm{e}^{-s/\alpha(x\cosh\tau+i(z-h)\sinh\tau)}\,\mathrm{d}\tau, \qquad t > \frac{r}{\alpha} \tag{7.8}$$

we find the operator equation

$$U = L^{-1}\int \mathrm{d}\tau \tag{7.9}$$

holds when both sides of (7.9) are applied to $\tilde\phi$ of equation (7.5). It is evident that the application of the operator U to $\tilde\phi$ in (7.6) is simpler than the application of the operator $L^{-1}\int \mathrm{d}\tau$ in equation (7.8). Further examples of the use of the differential transform for solving problems of interest in engineering are given in [8, 9, 12–19].

8. SOLUTION OF GARVIN'S PROBLEM BY MEANS OF THE DIFFERENTIAL TRANSFORM

Using algebraic functions of the parameter τ, in preference to the hyperbolic functions of Diagram 3.1, consider the plane wave

$$\tilde\phi_0 = s^{-1}A_0(\tau)\,\mathrm{e}^{-s/\alpha(\tau|x|+F_a(\tau)|z-h|)} \tag{8.1}$$

where

$$A_0(\tau) = (\tau^2-1)^{-1/2} \quad\text{and}\quad F_a(\tau) = (1-\tau^2)^{1/2}. \tag{8.2}$$

$\tilde\phi_0$ has an appropriate form for the application of the differential transform, and the real part of its differential transform gives Garvin's line source function,

$$\phi_0 = \begin{cases} \cosh^{-1}\dfrac{\alpha t}{r_0}, & |t| > \dfrac{r_0}{\alpha} \\ 0, & |t| < \dfrac{r_0}{\alpha} \end{cases}, \quad r_0^2 = x^2 + (z-h)^2. \tag{8.3}$$

Instead of solving Garvin's problem directly for the source ϕ_0 of equation (8.3), we shall first solve it for the simpler source $\tilde{\phi}_0$ of equation (8.1). The application of the operator ReU to the solution associated with the source $\tilde{\phi}_0$ gives a solution associated with ϕ_0. The simpler source, $\tilde{\phi}_0$, is a plane wave, the solution for which is well known. Thus using $\tilde{\phi}_0$ as a source for the reduced problem, equation (3.1), a reduced plane wave solution is obtained, which becomes a solution to Garvin's problem after taking the real part of its differential transform. As we know from the theory of plane waves, the solution fo the reduced problem, formed by the reduced wave equations (8.5) and boundary conditions (8.6) below, is the pair $\tilde{\phi}$ and $\tilde{\psi}$ where

$$\tilde{\phi} = \tilde{\phi}_0 + \tilde{\phi}_1, \qquad \tilde{\psi} = \tilde{\psi}_1. \tag{8.4}$$

In (8.4), $\tilde{\phi}_0$ represents the primary field, i.e. the reduced wave field that would exist if the boundary were absent, while $\tilde{\phi}_1$ and $\tilde{\psi}_1$ represent the secondary fields accounting for the presence of the boundary. In equation (8.4) $\tilde{\phi}_0$ is a given solution of the first equation in (8.5) below. $\tilde{\phi}_1$ and $\tilde{\psi}_1$ in (8.4) are unknown solutions of (8.5) that are introduced so as to satisfy the boundary conditions (8.6). Thus the functions $\tilde{\phi}$ and $\tilde{\psi}$ of (8.4) must satisfy the reduced wave equations

$$\nabla^2\tilde{\phi} = \frac{s^2}{\alpha^2}\tilde{\phi} \qquad \nabla^2\tilde{\psi} = \frac{s^2}{\beta^2}\tilde{\psi} \tag{8.5}$$

and the boundary conditions at $z = 0$

$$\lambda\nabla^2\tilde{\phi} + 2\mu\left[\frac{\partial^2\tilde{\phi}}{\partial z^2} + \frac{\partial^2\tilde{\psi}}{\partial x \partial z}\right] = 0 \qquad \mu\left[2\frac{\partial^2\tilde{\phi}}{\partial x\partial z} + \frac{\partial^2\tilde{\psi}}{\partial x^2} - \frac{\partial^2\tilde{\psi}}{\partial z^2}\right] = 0. \tag{8.6}$$

Equations (8.5) and (8.6) are obtained by applying the differential transform to equations (2.2) and (2.4) respectively, and using the fundamental properties in (6.2). The method of solving the boundary value problem of equations (8.5) and (8.6) for plane waves is well known. Since $\tilde{\phi}_0$ is a reduced plane wave, equation (8.1) suggests that $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\tilde{\psi}_1$ have the form:

$$\begin{aligned} \tilde{\phi}_0 &= s^{-1}A_0(\tau)\, e^{-s/\alpha(\tau|x| + F_a(\tau)|z-h|)} \\ \tilde{\phi}_1 &= s^{-1}A_1(\tau)\, e^{-s/\alpha(\tau|x| + F_a(\tau)(z+h))} \\ \tilde{\psi}_1 &= s^{-1}B_1(\tau)\, e^{-s/\alpha(\tau|x| + F_b(\tau)z + F_a(\tau)h)} \end{aligned} \tag{8.7}$$

where $F_a(\tau)$ and $F_b(\tau)$ are determined by the reduced wave equations (8.5),

$$F_a(\tau) = (1-\tau^2)^{1/2}, \qquad F_b(\tau) = \left(\frac{\alpha^2}{\beta^2} - \tau^2\right)^{1/2}. \tag{8.8}$$

$A_0(\tau)$ is taken as

$$A_0(\tau) = (\tau^2 - 1)^{-1/2} \tag{8.9}$$

in order that $\phi_0 = ReU\{\tilde{\phi}_0\}$ is Garvin's source (8.3). In choosing the signs of the functions within the exponentials of equations (8.7), a number of options are available. For $\tilde{\phi}_0$ absolute values are used in order to produce symmetry about the $x = 0$ and $z = h$ axes.

For $\tilde{\phi}_1$ and $\tilde{\psi}_1$ absolute values are used to produce symmetry about the $x = 0$ axis. Positive functions, $+F_a(\tau)$ and $+F_b(\tau)$, are used since for physical reasons these waves may be considered as arising from images located in the half-space $z > 0$. A_1 and B_1, determined by the boundary conditions (8.6), are

$$A_1(\tau) = A_0(\tau)\left[\frac{4\mu\tau^2 F_a F_b + (\lambda + 2\mu(1-\tau^2))(\tau^2 - F_b^2)}{4\mu\tau^2 F_a F_b - (\lambda + 2\mu(1-\tau^2))(\tau^2 - F_b^2)}\right]$$

$$B_1(\tau) = A_0(\tau)\left[\frac{4\tau F_a(\lambda + 2\mu(1-\tau^2))}{4\mu\tau^2 F_a F_b - (\lambda + 2\mu(1-\tau^2))(\tau^2 - F_b^2)}\right]. \tag{8.10}$$

Thus, equations (8.4), (8.7)–(8.10) provide a solution to a reduced Garvin's boundary value problem. It is associated with the reduced plane wave $\tilde{\phi}_0$. The solution is obtained by a standard well-known method for solving problems in plane waves.

To obtain a solution of Garvin's problem associated with the impulsive line source ϕ_0 it remains to apply the operator ReU to the solution associated with $\tilde{\phi}_0$. The displacement field can then be found by means of appropriate differentiations of the potentials as explained in (2.1).

Since differentiations can be interchanged with U, equation (6.2), the resulting displacement field (u, w) is

$$\begin{aligned} u &= \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z} = ReU\,\frac{\partial}{\partial x}\{\tilde{\phi}_0 + \tilde{\phi}_1\} - ReU\,\frac{\partial}{\partial z}\{\tilde{\psi}_1\} \\ &= -\frac{1}{\alpha}\,ReU\{\operatorname{sign}(x)\tau\tilde{\phi}_0 + \operatorname{sign}(x)\tau\tilde{\phi}_1 + F_b(\tau)\tilde{\psi}_1\} \\ &= -\frac{1}{\alpha}\,Re\{\operatorname{sign}(x)\tau_0^* A_0(\tau_0^*)\dot{\tau}_0^* + \operatorname{sign}(x)\tau_1^* A_1(\tau_1^*)\dot{\tau}_1^* + F_b(\tau_2^*)B_1(\tau_2^*)\dot{\tau}_2^*\} \end{aligned} \tag{8.11}$$

$$\begin{aligned} w &= \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x} = ReU\,\frac{\partial}{\partial z}\{\tilde{\phi}_0 + \tilde{\phi}_1\} + ReU\,\frac{\partial}{\partial x}\{\tilde{\psi}_1\} \\ &= -\frac{1}{\alpha}\,ReU\{\operatorname{sign}(z-h)F_a(\tau)\tilde{\phi}_0 + F_a(\tau)\tilde{\phi}_1 + \operatorname{sign}(x)\tau\tilde{\psi}_1\} \\ &= -\frac{1}{\alpha}\,Re\{\operatorname{sign}(z-h)F_a(\tau_0^*)A_0(\tau_0^*)\dot{\tau}_0^* + F_a(\tau_1^*)A_1(\tau_1^*)\dot{\tau}_1^* + \operatorname{sign}(x)\tau_2^* B_2(\tau_2^*)\dot{\tau}_2^*\} \end{aligned} \tag{8.12}$$

τ_0^* is a root of the quadratic equation:

$$r_0^2\tau_0^{*2} - 2x\alpha t\tau_0^* + \alpha^2 t^2 - (z-h)^2 = 0 \tag{8.13}$$

found by inverting the determining function

$$t^*(x, z, \tau) = \frac{1}{\alpha}(\tau|x| + Fa(\tau)|z-h|) \tag{8.14}$$

of the potential $\tilde{\phi}_0$. Similarly τ_1^* is a root of the quadratic equation

$$r_1^2\tau_1^{*2} - 2x\alpha t\tau_1^* + \alpha^2 t^2 - (z+h)^2 = 0 \tag{8.15}$$

where

$$r_1^2 = x^2 + (z+h)^2. \tag{8.16}$$

τ_2^* is a root of the quartic equation

$$C_4\tau_2^{*4} + C_3\tau_2^{*3} + C_2\tau_2^{*2} + C_1\tau_2^* + C_0 = 0 \tag{8.17}$$

where for $k^2 = \alpha^2/\beta^2$

$$\begin{aligned}
C_0 &= (\alpha^2t^2 - k^2z^2 - h^2)^2 - 4h^2z^2k^2 \\
C_1 &= -4\alpha tx(\alpha^2t^2 - k^2z^2 - h^2) \\
C_2 &= 4\alpha^2t^2x^2 + 2(x^2 + z^2 + h^2)(\alpha^2t^2 - k^2z^2 - h^2) + 4h^2z^2(1 + k^2) \\
C_3 &= -4\alpha tx(x^2 + z^2 + h^2) \\
C_4 &= (x^2 + z^2 + h^2)^2 - 4h^2z^2.
\end{aligned} \tag{8.18}$$

The quartic equation (8.17) is found by inverting the determining function

$$t^*(x, z, \tau) = \frac{1}{\alpha}(\tau|x| + F_b(\tau)z + F_a(\tau)h) \tag{8.19}$$

of the potential $\tilde{\psi}_1$. $\dot{\tau}_0^*$, $\dot{\tau}_1^*$ and $\dot{\tau}_2^*$ are obtained by differentiating equations (8.13), (8.15) and (8.17) respectively with respect to t.

$$\dot{\tau}_0^* = \frac{x\alpha\tau_0^* - \alpha^2 t}{r_0^2\tau_0^* - x\alpha t} \tag{8.20}$$

$$\dot{\tau}_1^* = \frac{x\alpha\tau_1^* - \alpha^2 t}{r_1^2\tau_1^* - x\alpha t} \tag{8.21}$$

$$\dot{\tau}_2^* = -\frac{\dot{C}_3\tau_2^{*3} + \dot{C}_2\tau_2^{*2} + \dot{C}_1\tau_2^* + \dot{C}_0}{4C_4\tau_2^{*3} + 3C_3\tau_2^{*2} + 2C_2\tau_2^* + C_1} \tag{8.22}$$

where $\dot{C}_i = \partial C_i/\partial t$,

$$\begin{aligned}
\dot{C}_0 &= 4\alpha^2t(\alpha^2t^2 - k^2z^2 - h^2) \\
\dot{C}_1 &= -4\alpha x(3\alpha^2t^2 - k^2z^2 - h^2) \\
\dot{C}_2 &= 4\alpha^2t(3x^2 + z^2 + h^2) \\
\dot{C}_3 &= -4\alpha x(x^2 + z^2 + h^2) \\
\dot{C}_4 &= 0.
\end{aligned} \tag{8.23}$$

Since each of the solutions τ_2^* of the quartic equation (8.17) yields a solution to Garvin's problem, the solution is not unique. A unique solution is singled out, as discussed in the next section, by requiring the solution to introduce no new singularities into the solid; the only singularity allowed is that due to the line source.

9. NUMERICAL RESULTS

Numerical evaluation of the displacement reveals the unique solution to Garvin's problem. At a given point (x, z) and a given time t the horizontal and vertical displacements, u and w, may be computed by making use of equations (8.11) and (8.12). Varying t over some interval, graphs may be plotted of displacement versus time.

Excluding surface waves, three pulses are expected in the time interval $(0, \infty)$. According to ray theory, the first to arrive is the direct P-pulse and its arrival time t_p is

$$t_p = \frac{1}{\alpha}(x^2 + (z - h)^2)^{1/2} = \frac{r_0}{\alpha}. \tag{9.1}$$

Next to arrive is the reflected P-pulse with arrival time t_{pp}

$$t_{pp} = \frac{1}{\alpha}(x^2 + (z + h)^2)^{1/2} = \frac{r_1}{\alpha}. \tag{9.2}$$

Finally the reflected S-pulse arrives at time t_{ps}

$$t_{ps} = \frac{1}{\alpha}(h^2 + x_1^2)^{1/2} + \frac{1}{\beta}(z^2 + (x - x_1)^2)^{1/2}, \tag{9.3}$$

where x_1 is the real root of the quartic equation

$$(\alpha^2 - \beta^2)x_1^4 + 2x(\beta^2 - \alpha^2)x_1^3 + (\alpha^2(x^2 + h^2) - \beta^2(x^2 + z^2))x_1^2 - 2\alpha^2h^2xx_1 + \alpha^2h^2x^2 = 0 \tag{9.4}$$

uniquely determined by the inequality $0 < x_1 < x$.

The functions used in the evaluation of the displacement components u and w are τ_0^*, $\dot{\tau}_0^*$, A_0, τ_1^*, $\dot{\tau}_1^*$, A_1, τ_2^*, $\dot{\tau}_2^*$ and B_1. In general these are all complex functions of a complex root of either a quadratic or a quartic equation. The question is, which root to use in order to obtain the right unique solution to the geophysical problem of Garvin.

When quadratic equations are involved the question is immaterial, since the two roots are complex conjugates and both give the same result.

When the quartic equation is involved in the displacement calculation, the situation is more complex. The four roots form two pairs of complex conjugates, one of which must be rejected. One pair results in the displacement having pulses with arrival times in agreement with ray theory, equations (9.1)–(9.3), while the other pair introduce new singularities into the elastic half-space under consideration and therefore must be rejected. The following plots of horizontal displacement due to the PS pulse, in Figs. 9.1 and 9.2, demonstrate this.

The arrows denoted by P, PP and PS in Figs. 9.1–9.7 indicate the arrival times of the P-, PP- and PS-pulses respectively, as evaluated in ray theory. The agreement of the ray theory with the analytic one is evident.

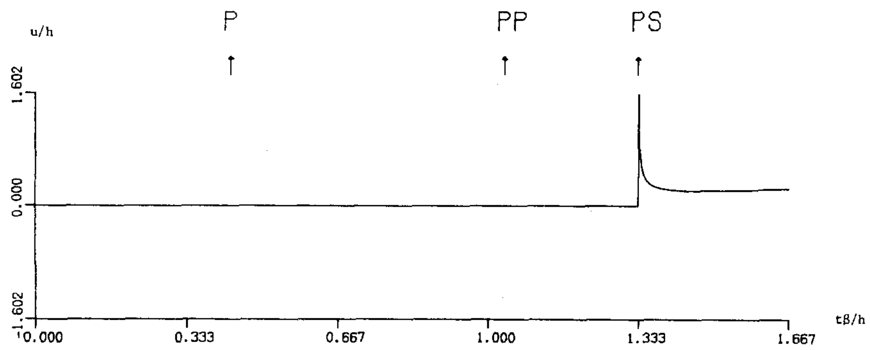

Fig. 9.1. Horizontal displacement due to the PS pulse, using a root of the quartic equation which results in the PS pulse having the arrival time expected from ray theory. The parameters are; $(0, h)$ for the point source, $(2h/3, 2h/3)$ for the point of observation and $\sqrt{3}$ for α/β.

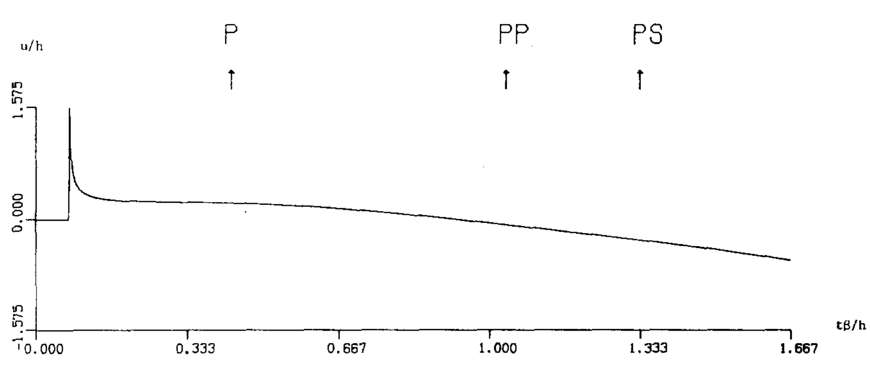

Fig. 9.2. Horizontal displacement due to the PS pulse, using a root of the quartic equation which results in the PS pulse displaying a new singularity. The parameters are; $(0, h)$ for the point source, $(2h/3, 2h/3)$ for the point of observation and $\sqrt{3}$ for α/β.

The source ϕ_0 may be considered "impure" in the following sense: A plot of horizontal displacement due to the mathematical representation of the physical source,

$$\phi_0 = Re\partial_t^{-1}\left(t^2 - \frac{r^2}{\alpha^2}\right)^{-1/2},$$

equation (3.5), shows not only a pulse at $t = r_0/\alpha$ as expected physically, but also what we choose to call an "anti-pulse" at $t = -r_0/\alpha$ contrary to the physical situation. Formally a pulse is a function which is zero for all t before a certain arrival time t_0, whereas an anti-pulse is a function which is zero for all t after a certain arrival time t_0.

Due to this impurity, it was found that for some observer placements the displacement due to the PS wave exhibits not only a pulse with the expected arrival time, t_{ps}, but also an anti-pulse with arrival time less than t_{ps}. The anti-pulse is recognized by its form, Fig. 9.3. The anti-pulse has no effect on displacement for $t > t_{ps}$. It may be suppressed artificially by zeroising the displacement due to the PS wave when $t < t_{ps}$. Figure 9.4 demonstrates the appearance of the anti-pulse.

We conclude with plots of displacement components due to P, PP and PS pulses, which present the unique solution of Garvin's problem obtained by the method of the differential transform.

The solution of Garvin's problem, presented numerically in Figs. 9.5 and 9.6, is identical to that obtained by means of Cagniard's method. However, this solution is obtained with no reference to integral considerations.

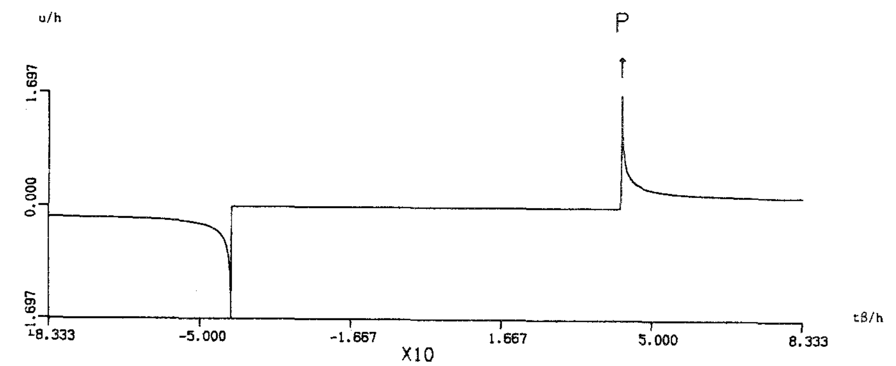

Fig. 9.3. Horizontal displacement due to the source function showing not only a pulse at $t = r_0/\alpha$, but also an "anti-pulse" at $t = -r_0/\alpha$. The parameters are; $(0, h)$ for the point source, $(2h/3, 2h/3)$ for the point of observation and $\sqrt{3}$ for α/β.

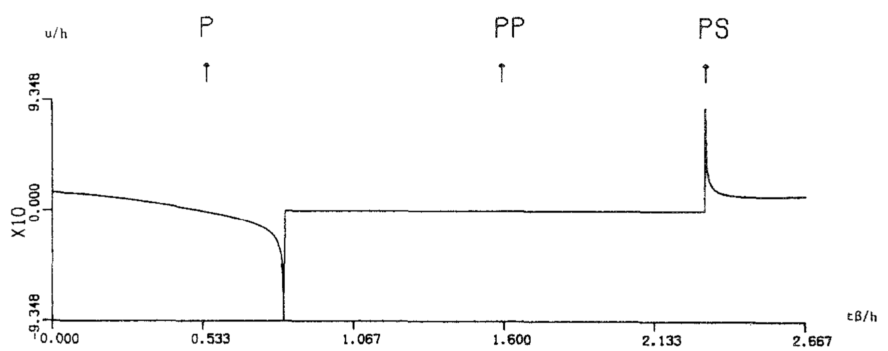

Fig. 9.4. Horizontal displacement due to the PS pulse, showing the effect of the source's "anti-pulse". The parameters are; $(0, h)$ for the point source, $(5h/3, 2h/3)$ for the point of observation and $\sqrt{3}$ for α/β.

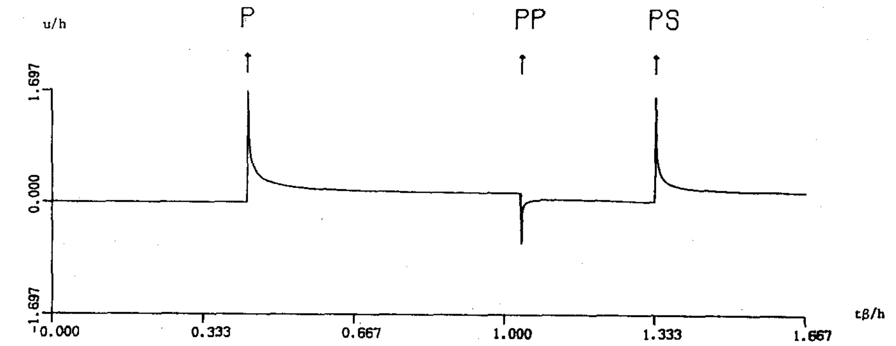

Fig. 9.5. Horizontal displacement due to P, PP and PS pulses, using correct roots and eliminating any "anti-pulses" that may occur. The parameters are; $(0, h)$ for the point source, $(2h/3, 2h/3)$ for the point of observation and $\sqrt{3}$ for α/β.

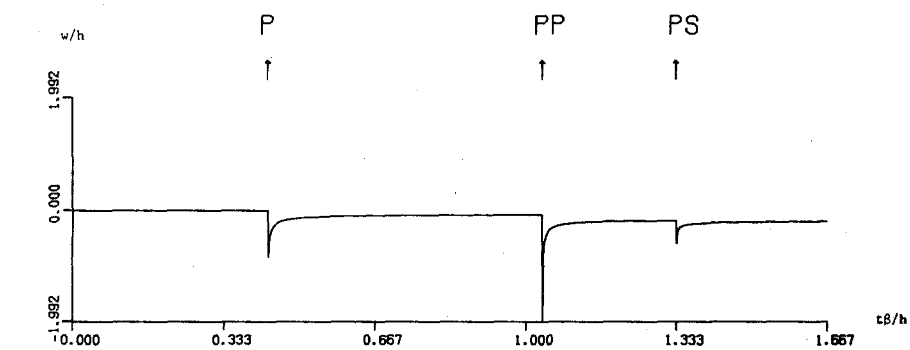

Fig. 9.6. Vertical displacement due to P, PP and PS pulses, using correct roots and eliminating any "anti-pulses" that may occur. The parameters are; $(0, h)$ for the point source, $(2h/3, 2h/3)$ for the point of observation and $\sqrt{3}$ for α/β.

10. CONCLUSION

Two methods for solving Garvin's problem were displayed. One method, Cagniard's method, utilizes the integral transform representation (7.8). The other method, that of the differential transform, utilizes the differential transform representation (7.7). It was demonstrated that the method of the differential transform is simpler than that of Cagniard enabling one to avoid the integral considerations involved in Cagniard's method.

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